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BY

GEORGE B. DANTZIG

TECHNICAL REPORT SOL 77-17

JUNE 1977

Systems Optimization Laboratory
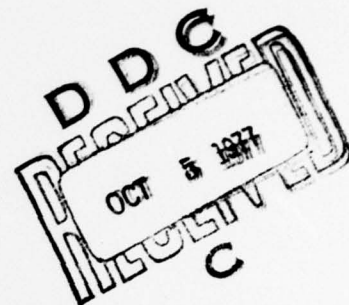
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DISCOVERING HIDDEN TOTALLY LEONTIEF SUBSTITUTION SYSTEMS

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George B. Dantzig

A <u>Totally Leontief Square System</u> $Lu = b$ is defined by the following properties:

(a) Every column of L has at most one positive element; $b > 0$; L is $m \times m$.

(b) $Lu = b$ has a solution $u \geq 0$.

It is easy to see that every row of L has at least one positive element -- indeed there are exactly m positive elements, each in a different column. Other properties are: L is non-singular, $L^{-1} \geq 0$ and $Lu = b$, $\pi L = c$ yields $u > 0$, $\pi > 0$ for all b ?, $c > 0$. See [4].

A <u>Totally Leontief Substitution System</u> $Ax = \quad$ defined by the following properties:

(c) Every column $A(*,j)$ of A has at most one positive component; $b > 0$; A is $m \times n$.

(d) The linear program; Find Max z:

(1) $$Ax = b , \quad x \geq 0, ex = z ,$$

where $e = (1, 1, \ldots, 1)$, is feasible and Max **z** is finite.

---

*Arthur F. Veinott, Jr. has made several suggestions that are incorporated into the algorithm and proofs presented, see [7].

1

Other properties are that every column of A has exactly one

positive element; every row of A has one or more positive elements;

every subset of m columns with a positive element in each row forms

a feasible basis. All basic feasible solutions are non-degenerate.

Main references are [3], [7], [8], and [9].

Substitution Classes: Associated with any feasible basis B

to a general linear program is the set S_i of column indices j such that

$A(*,j)$ if substituted for $B(*,i)$ forms a feasible basis. S_i is

non-empty for each i since it includes j: $A(*,j) = B(*,i)$. In

general S_i depends on B. For the Leontief substitution case, however,

it is independent of the choice of B, since the substitution class

S_i clearly consists of all j that have a positive element in the

same row. It is convenient to have S_i correspond to all columns j

whose positive element is in row i. We shall refer to the latter as

"row-standardized" substitution classes.

A system

(1) $\bar{A}x = \bar{b}$

where $\bar{A}$ is m × n is a "hidden" totally Leontief substitution system

if there exists a nonsingular matrix R such that

(2) $(R\bar{A})x = (R\bar{b})$

is a totally Leontief substitution system.

2

The Problem. Given a linear system $\bar{A}x = \bar{b}$ find a nonsingular matrix R if it exists such that $R\bar{A}x = R\bar{b}$ is a totally Leontief substitution system.

Constructive Solution.

Step 1: Solve the linear program: Find Max z

(2) $\qquad \bar{A}x = \bar{b}$, $\quad x \geq 0$, ex = z (Max) .

Terminate if infeasible or unbounded, or if the optimal basic feasible solution is degenerate. Let $\hat{B}$ be the optimal feasible basis.

Step 2: Determine substitution classes S_1, S_2, ..., S_m with respect to $\hat{B}$. Terminate if there is ambiguity about which substitution class a column belongs to.

Step 3: For each i, for i = (1, ..., m) determine multipliers R(i, *) such that

$$R(i,*)\ \bar{A}(*,j) \geq 1 , \qquad\qquad \text{if } j \in S_i$$

$$\leq 0 , \qquad\qquad \text{if } j \notin S_i$$

$$R(i,*)\bar{b} \geq 1 .$$

Terminate if for any i = (1, ..., m) the above linear inequality system is infeasible. If not terminate, the matrix R whose rows are R(i, *) is the solution.

Proof Step 1: If the original system (2) is infeasible or unbounded, i.e., an optimal solution of the linear program (2) fails to exist, then the same is true for every linear transform using non-singular R, contradicting the existence of an R for (2) such that (d) holds.

3

Proof Step 2: The only way there can be difficulty in determining which columns an entering column replaces in the basis is the unbounded case (ruled out on Step 1) or there is a tie in the ratio test for selecting a pivot in the simplex method. The latter implies that the feasible basic feasible solution obtained after the substitution is degenerate. This is ruled out since all basic feasible solutions are non-degenerate in a Leontief substitution system.

Proof Step 3: Note that feasibility and the adjacency properties of feasible bases that differ by one column are invariant under non-singular linear transforms. After transformation by $R(i, *)$, assuming R exists, row i of $R\bar{A}$ should have $+$ components corresponding to substitution class S_k. We can permute the rows of R so that S_i corresponds to the "row standardized" substitution class associated with i of $R\bar{A}$, namely $R(i, *) A(*, j) > 0$ for $j \in S_i$ and $R(i, *) A(*, j) \leq 0$ otherwise; we also have $R(i, *) \cdot \bar{b} > 0$ for $i = (1, \ldots, m)$. Step III can be solved by applying Phase I of the simplex method to a linear program.

4

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In earlier work by R. Saigal, necessary and sufficient conditions were stated for the equivalence of a given linear program with a totally Leontief substitution system. In this paper, we present a constructive procedure for determining the existence and evaluating (when it does exist) the non-singular matrix that transforms the L.P. into a member of the substitution class.

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